▨ **計算題**（共 100 分）

1. (1) 設 θ 為第二象限角，且 $\csc\theta = 3$，試求 $\tan2\theta$ 之值．（10 分）

   (2) 設 θ 為第三象限角，且 $\sin\theta + \cos\theta = -\sqrt{2}$，試求 θ 之值．（15 分）

   解：(1) 由 θ 為第二象限角，且 $\csc\theta = 3$

$$\Rightarrow \sin\theta = \frac{1}{3} , \cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-(\frac{1}{3})^2} = \frac{-2\sqrt{2}}{3}$$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{3}}{\frac{-2\sqrt{2}}{3}} = -\frac{\sqrt{2}}{4}$$

$$\therefore \tan2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times(-\frac{\sqrt{2}}{4})}{1-(-\frac{\sqrt{2}}{4})^2} = \frac{-\frac{\sqrt{2}}{2}}{\frac{7}{8}} = -\frac{4\sqrt{2}}{7} \text{ 即為所求}$$

(2) $\sin\theta + \cos\theta = -\sqrt{2} \Rightarrow (\sin\theta + \cos\theta)^2 = (-\sqrt{2})^2$

$\Rightarrow 1 + 2\sin\theta\cos\theta = 2 \Rightarrow 1 + \sin2\theta = 2 \Rightarrow \sin2\theta = 1$

又 $\pi < \theta < \dfrac{3\pi}{2} \Rightarrow 2\pi < 2\theta < 3\pi$

$\therefore 2\theta = 2\pi + \dfrac{\pi}{2} = \dfrac{5\pi}{2} = 450° $，故 $\theta = \dfrac{5\pi}{4} = 225°$ 即為所求

2. 設 $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$，且 $\sin2\theta = \dfrac{3}{5}$，試求 $\cos\theta$ 之值．（25 分）

   解：$\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < 2\theta < \pi$，即 θ 在第一象限，2θ 在第二象限

由 $\sin2\theta = \dfrac{3}{5} \Rightarrow \cos2\theta = -\sqrt{1-\sin^2 2\theta} = -\sqrt{1-(\dfrac{3}{5})^2} = -\dfrac{4}{5}$

又由 $\cos2\theta = 2\cos^2\theta - 1 \Rightarrow 2\cos^2\theta - 1 = -\dfrac{4}{5} \Rightarrow \cos^2\theta = \dfrac{1}{10}$

$\therefore \cos\theta = +\sqrt{\dfrac{1}{10}} = \dfrac{1}{\sqrt{10}}$ 即為所求

3. 設 $0 < \theta < \dfrac{\pi}{2}$ ，且 $\cot 2\theta = \dfrac{3}{4}$ ，試求 $\cos\theta$ 之值 ．（25 分）

解：

$0 < \theta < \dfrac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$ ，又 $\cot 2\theta = \dfrac{3}{4} > 0$ ，故 2θ 在第一象限

由 $\cot 2\theta = \dfrac{3}{4} = \dfrac{x}{y} \Rightarrow \sin 2\theta = \dfrac{4}{5}$ ，$\cos 2\theta = \dfrac{3}{5}$

$\therefore \cos\theta = \sqrt{\dfrac{1 + \cos 2\theta}{2}} = \sqrt{\dfrac{1 + \dfrac{3}{5}}{2}} = \sqrt{\dfrac{4}{5}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$ 即為所求

4. 設 $\dfrac{3\pi}{2} < \theta < 2\pi$ ，且 $\tan\theta = -\dfrac{4}{3}$ ，試求 $\sin\dfrac{3\theta}{2}$ 之值 ．（25 分）

解：

由 θ 在第四象限，且 $\tan\theta = \dfrac{-4}{3} = \dfrac{y}{x} \Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$

$\Rightarrow \sin\theta = \dfrac{y}{r} = \dfrac{-4}{5}$ ，$\cos\theta = \dfrac{x}{r} = \dfrac{3}{5}$

又 $\dfrac{3\pi}{2} < \theta < 2\pi \Rightarrow \dfrac{3\pi}{4} < \dfrac{\theta}{2} < \pi$ ，即 $\dfrac{\theta}{2}$ 在第二象限

$\Rightarrow \sin\dfrac{\theta}{2} = +\sqrt{\dfrac{1 - \cos\theta}{2}} = \sqrt{\dfrac{1 - \dfrac{3}{5}}{2}} = \dfrac{1}{\sqrt{5}}$

$\therefore \sin\dfrac{3\theta}{2} = \sin\left(3 \times \dfrac{\theta}{2}\right) = 3\sin\dfrac{\theta}{2} - 4\sin^3\dfrac{\theta}{2} = 3\left(\dfrac{1}{\sqrt{5}}\right) - 4\left(\dfrac{1}{\sqrt{5}}\right)^3$

$\qquad = \dfrac{3}{\sqrt{5}} - \dfrac{4}{5\sqrt{5}} = \dfrac{15\sqrt{5} - 4\sqrt{5}}{25} = \dfrac{11\sqrt{5}}{25}$ 即為所求