高雄市明誠中學 高一數學平時測驗 日期：99.06.25					
範圍	3-3 倍角與半角公式	班級		姓名	
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一、填充題 （ 每題 100 分 ）

1. 設 $f(x)=8x^3+4x^2-6x-2$，則以 $x-\cos 15°$ 除 $f(x)$ 的餘式為____________ .

解答 $\sqrt{3}+\sqrt{2}$

解析 由餘式定理知餘式為

$$f(\cos 15°)=8(\cos 15°)^3+4(\cos 15°)^2-6(\cos 15°)-2$$

$$=2\left[4\cos^3 15°-3\cos 15°\right]+4(\cos 15°)^2-2$$

$$=2\cos 45°+4(\cos 15°)^2-2=2\cdot\frac{\sqrt{2}}{2}+4\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2-2=\sqrt{3}+\sqrt{2}\ .$$

2. $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=$____________ .

解答 $\frac{1}{8}$

解析 原式 $=\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\left(-\cos\frac{4\pi}{7}\right)=-\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$$=-\frac{1}{2^3\sin\frac{\pi}{7}}\left(2^3\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2}{7}\pi\cos\frac{4}{7}\pi\right)$$

$$=-\frac{1}{8\sin\frac{\pi}{7}}\left(\sin\frac{8\pi}{7}\right)=-\frac{1}{8\sin\frac{\pi}{7}}\left(-\sin\frac{\pi}{7}\right)=\frac{1}{8}\ .$$

3. 試求 $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$ 的值____________ .

解答 $\frac{3}{2}$

解析 原式 $=\cos^4\frac{\pi}{8}+\cos^4\frac{3}{8}\pi+\cos^4\left(\pi-\frac{3}{8}\pi\right)+\cos^4\left(\pi-\frac{\pi}{8}\right)$

$$=2\left[\cos^4\frac{\pi}{8}+\cos^4\frac{3}{8}\pi\right]=2\left[\left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1+\cos\frac{3}{4}\pi}{2}\right)^2\right]$$

$$=2\cdot\left[\left(\frac{1+\frac{\sqrt{2}}{2}}{2}\right)^2+\left(\frac{1-\frac{\sqrt{2}}{2}}{2}\right)^2\right]=2\cdot\left(\frac{3}{4}\right)=\frac{3}{2}\ .$$

4.θ是第二象限角，若$\sin 2\theta=-\frac{4}{5}$，求$\sin\theta=$____________ .

解答 $\frac{1}{\sqrt{5}}$或$\frac{2}{\sqrt{5}}$

解析 $\because\theta$為第二象限角且$\sin 2\theta=-\frac{4}{5}<0\Rightarrow\cos 2\theta=+\frac{3}{5}$或$\cos 2\theta=-\frac{3}{5}$

$$\sin\theta=+\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-\frac{3}{5}}{2}}=\frac{1}{\sqrt{5}}\text{ 或}\sin\theta=+\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}=\frac{2}{\sqrt{5}}\ .$$

5. 在$\triangle ABC$，已知$\cos A=-\frac{4}{5}$，求$\tan\frac{A}{2}=$____________ .

解答 3

解析 令$\tan\frac{A}{2}=t$，$\cos A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}=\frac{1-t^2}{1+t^2}=-\frac{4}{5}\Rightarrow t^2=9\Rightarrow t=\pm 3$（取正），即$\tan\frac{A}{2}=3$.

6. 設$\sin\theta=\frac{8}{7}\cos\frac{\theta}{2}$，則$\cos\theta=$____________ .

解答 -1或$\frac{17}{49}$

解析 $\sin\theta=\frac{8}{7}\cos\frac{\theta}{2}\Rightarrow 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{8}{7}\cos\frac{\theta}{2}\Rightarrow\cos\frac{\theta}{2}\left(2\sin\frac{\theta}{2}-\frac{8}{7}\right)=0$

① $\cos\frac{\theta}{2}=0\Rightarrow\cos\theta=2\cos^2\frac{\theta}{2}-1=-1$

② $2\sin\frac{\theta}{2}=\frac{8}{7}\Rightarrow\sin\frac{\theta}{2}=\frac{4}{7}\Rightarrow\cos\theta=1-2\sin^2\frac{\theta}{2}=1-2\times\frac{16}{49}=\frac{17}{49}$

由①②$\Rightarrow\cos\theta=-1$或$\frac{17}{49}$.

7.$8\sqrt{3}\cos^3 20^\circ+6\sin 20^\circ-6\sqrt{3}\cos 20^\circ-8\sin^3 20^\circ=$____________ .

解答 $2\sqrt{3}$

解析 原式$=8\sqrt{3}\cos^3 20^\circ-6\sqrt{3}\cos 20^\circ+2\left(3\sin 20^\circ-4\sin^3 20^\circ\right)$

$$=2\sqrt{3}\left(4\cos^3 20^\circ-3\cos 20^\circ\right)+2\cdot\sin 60^\circ$$

$$=2\sqrt{3}\cdot\cos 60^\circ+2\cdot\sin 60^\circ=2\sqrt{3}\cdot\frac{1}{2}+2\cdot\frac{\sqrt{3}}{2}=2\sqrt{3}$$.

8. $2+\sqrt{3}$ 為 $x^2-(\tan\theta+\cot\theta)x+1=0$ 的一根，則 $\sin 2\theta=$____________ .

解答 $\frac{1}{2}$

解析 設另一根 β ，則 $(2+\sqrt{3})\cdot\beta=1\Rightarrow\beta=2-\sqrt{3}$

又兩根和 $=(2+\sqrt{3})+(2-\sqrt{3})=\tan\theta+\cot\theta\Rightarrow 4=\frac{1}{\sin\theta\cos\theta}\Rightarrow\sin\theta\cos\theta=\frac{1}{4}$

$\therefore\sin 2\theta=2\sin\theta\cos\theta=\frac{1}{2}$.

9. 設 $\frac{\pi}{4}<\theta<\frac{\pi}{2}$ 且 $\sin 2\theta=\frac{4}{5}$，則 $\sin\theta=$____________ .

解答 $\frac{2\sqrt{5}}{5}$

解析 $\frac{\pi}{4}<\theta<\frac{\pi}{2}\Rightarrow\frac{\pi}{2}<2\theta<\pi$ ，$\sin 2\theta=\frac{4}{5}\Rightarrow\cos 2\theta=-\frac{3}{5}$

$\therefore\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1+\frac{3}{5}}{2}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$.

10.設 $\sin x+\cos x=\frac{1}{2}$，則(1) $\sin 2x=$____________ . (2) $\sin 3x-\cos 3x=$____________ .

解答 $(1)-\frac{3}{4}$;$(2)-\frac{5}{4}$

解析 (1) $\sin x+\cos x=\frac{1}{2}$

兩邊平方 $\Rightarrow\sin^2 x+\cos^2 x+2\sin x\cos x=\frac{1}{4}\Rightarrow 2\sin x\cdot\cos x=-\frac{3}{4}$ ，　故 $\sin 2x=-\frac{3}{4}$

(2) $\sin 3x-\cos 3x=(3\sin x-4\sin^3 x)-(4\cos^3 x-3\cos x)$

$=-4(\sin^3 x+\cos^3 x)+3(\sin x+\cos x)$

$=-4\left[(\sin x+\cos x)^3-3\sin x\cos x(\sin x+\cos x)\right]+3(\sin x+\cos x)$

$=-4\left[\left(\frac{1}{2}\right)^3-3\left(-\frac{3}{8}\right)\left(\frac{1}{2}\right)\right]+3\left(\frac{1}{2}\right)=-\frac{5}{4}$.

11.設 $\sin\theta+3\cos\theta=0$，則 $\sin 2\theta=$____________ .

解答 $-\dfrac{3}{5}$

解析 $\sin\theta=-3\cos\theta\Rightarrow\dfrac{\sin\theta}{\cos\theta}=-3\Rightarrow\tan\theta=-3$；$\therefore\sin2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{-6}{1+9}=-\dfrac{3}{5}$.

12.若 $\pi<\theta<\dfrac{3}{2}\pi$， $\tan\theta=\dfrac{4}{3}$，求 $\sin(3\theta)\tan\left(\dfrac{\theta}{2}\right)=$____________ .

解答 $\dfrac{88}{125}$

解析 $\because\pi<\theta<\dfrac{3\pi}{2}$ 且 $\tan\theta=\dfrac{4}{3}$，$\therefore\sin\theta=-\dfrac{4}{5}\Rightarrow\sin3\theta=3\sin\theta-4\sin^3\theta=-\dfrac{12}{5}-4\cdot\left(-\dfrac{64}{125}\right)=-\dfrac{44}{125}$

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{4}{3}\Rightarrow2\tan^2\frac{\theta}{2}+3\tan\frac{\theta}{2}-2=0$$

$\Rightarrow\left(2\tan\dfrac{\theta}{2}-1\right)\left(\tan\dfrac{\theta}{2}+2\right)=0$，$\therefore\tan\dfrac{\theta}{2}=-2$ 或 $\dfrac{1}{2}$（不合$\because\dfrac{\pi}{2}<\dfrac{\theta}{2}<\dfrac{3}{4}\pi$）

$\therefore\sin3\theta\cdot\tan\dfrac{\theta}{2}=\left(-\dfrac{44}{125}\right)\times(-2)=\dfrac{88}{125}$.

13.求 $\tan22.5^\circ=$____________ .（化成最簡根式）

解答 $\sqrt{2}-1$

解析 $\tan22.5^\circ=\tan\dfrac{45^\circ}{2}=\dfrac{1-\cos45^\circ}{\sin45^\circ}=\dfrac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=\sqrt{2}-1$.

14.(1)若 $\tan\theta=-\dfrac{4}{3}$，則 $\sin2\theta=$______. (2)若 $-\pi<\theta<0$，$\tan\theta=-\dfrac{4}{3}$，則 $\sin\dfrac{\theta}{2}=$______，$\sin3\theta=$______.

解答 (1) $-\dfrac{24}{25}$;(2) $-\dfrac{1}{\sqrt{5}}$，$-\dfrac{44}{125}$

解析 (1) $\sin2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2\left(-\frac{4}{3}\right)}{1+\left(-\frac{4}{3}\right)^2}=\dfrac{-\frac{8}{3}}{1+\frac{16}{9}}=-\dfrac{24}{25}$.

(2)$\because-\pi<\theta<0$，且 $\tan\theta=-\dfrac{4}{3}$ $\therefore\theta$ 為第四象限角

又 $\cos\theta=\dfrac{3}{5}$， $\sin\dfrac{\theta}{2}=-\sqrt{\dfrac{1-\cos\theta}{2}}=-\sqrt{\dfrac{1-\frac{3}{5}}{2}}=-\dfrac{1}{\sqrt{5}}$ ，

$\sin\theta=-\dfrac{4}{5}\Rightarrow\sin3\theta=3\sin\theta-4\sin^3\theta=3\left(-\dfrac{4}{5}\right)-4\left(-\dfrac{4}{5}\right)^3=-\dfrac{44}{125}$.

15.$\frac{5}{2}\pi<\theta<3\pi$，若$\sin\theta=\frac{3}{5}$，求(1) $\tan 2\theta=$______，(2) $\tan\frac{\theta}{2}=$______.

解答 (1) $-\frac{24}{7}$;(2) 3

解析 (1)$\because\frac{5}{2}\pi<\theta<3\pi$ $\therefore\theta$ 為第二象限又$\sin\theta=\frac{3}{5}\Rightarrow$ $\tan\theta=-\frac{3}{4}$

$$\therefore\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\cdot\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}=-\frac{24}{7}.$$

$$(2)\because\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\Rightarrow-\frac{3}{4}=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\Rightarrow 3\tan^2\frac{\theta}{2}-3=8\tan\frac{\theta}{2}\Rightarrow 3\tan^2\frac{\theta}{2}-8\tan\frac{\theta}{2}-3=0$$

$$\Rightarrow\left(3\tan\frac{\theta}{2}+1\right)\left(\tan\frac{\theta}{2}-3\right)=0\quad\therefore\tan\frac{\theta}{2}=3\text{ 或 }-\frac{1}{3}$$

$\because\frac{5}{4}\pi<\frac{\theta}{2}<\frac{3}{2}\pi\Rightarrow\frac{\theta}{2}$ 為第三象限， $\therefore\tan\frac{\theta}{2}=3$.

16.設$\cos\theta$ 為 $4x^2+4x-3=0$ 的一根，則$\cos 3\theta=$______.

解答 -1

解析 $4x^2+4x-3=0\Rightarrow(2x+3)(2x-1)=0$ $\Rightarrow x=-\frac{3}{2}$（不合）或$x=\frac{1}{2}=\cos\theta$

$$\therefore\cos 3\theta=4\cos^3\theta-3\cos\theta=4\cdot\left(\frac{1}{2}\right)^3-3\cdot\frac{1}{2}=-1.$$

17.設$\tan\alpha$、$\tan\beta$ 為 $x^2-4x+2=0$ 的二根，求(1) $\tan(\alpha+\beta)=$______. (2) $\tan\frac{\alpha+\beta}{2}=$______.

解答 (1) -4;(2) $\frac{1\pm\sqrt{17}}{4}$

解析 由根與係數關係$\tan\alpha+\tan\beta=4$， $\tan\alpha\cdot\tan\beta=2$

$$(1)\ \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}=-4.$$

$$(2)\text{令 }\tan\frac{\alpha+\beta}{2}=t\quad\Rightarrow\tan(\alpha+\beta)=\tan\left(2\cdot\frac{\alpha+\beta}{2}\right)=\frac{2\tan\frac{\alpha+\beta}{2}}{1-\tan^2\frac{\alpha+\beta}{2}}$$

$$\Rightarrow-4=\frac{2t}{1-t^2}\Rightarrow 2t^2-t-2=0\quad\Rightarrow t=\frac{1\pm\sqrt{17}}{4}.$$

18.$0<\alpha<\frac{\pi}{2}$，則$\frac{\sqrt{2-2\sin\alpha}}{\sqrt{1+\cos\alpha}-\sqrt{1-\cos\alpha}}=$______.

解答 1

解析 $\sqrt{2-2\sin\alpha}=\sqrt{2}\sqrt{1-\sin\alpha}=\sqrt{2}\left|\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right|=\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)$

$$\sqrt{\frac{1+\cos\alpha}{2}}=\left|\cos\frac{\alpha}{2}\right|\Rightarrow\sqrt{1+\cos\alpha}=\sqrt{2}\left|\cos\frac{\alpha}{2}\right|$$

同理 $\sqrt{1-\cos\alpha}=\sqrt{2}\left|\sin\frac{\alpha}{2}\right|$

$\therefore$ 原式 $=\dfrac{\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)}{\sqrt{2}\cos\frac{\alpha}{2}-\sqrt{2}\sin\frac{\alpha}{2}}=1$.

19.設 $\cos\left(\frac{\pi}{4}-\theta\right)=k$，請用 k 表示 $\cos 4\theta=$____________ .

解答 $-8k^4+8k^2-1$

解析 $\cos\left(\frac{\pi}{4}-\theta\right)=k$

$$\therefore\cos\left[2\left(\frac{\pi}{4}-\theta\right)\right]=2\cos^2\left(\frac{\pi}{4}-\theta\right)-1\Rightarrow\cos\left(\frac{\pi}{2}-2\theta\right)=2\cos^2\left(\frac{\pi}{4}-\theta\right)-1\Rightarrow\sin 2\theta=2k^2-1$$

故 $\cos 4\theta=1-2\sin^2 2\theta=1-2\left(2k^2-1\right)^2=-8k^4+8k^2-1$.

20.已知 $2\cos\theta=a+\frac{1}{a}$，則 $2\cos 3\theta$ 以 a 表之為____________ .

解答 $a^3+\frac{1}{a^3}$

解析 $2\cos 3\theta=2\left(4\cos^3\theta-3\cos\theta\right)=8\cos^3\theta-6\cos\theta=\left(2\cos\theta\right)^3-3\left(2\cos\theta\right)$

$$=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)=a^3+\frac{1}{a^3}$$.

21.設 $t=\tan\frac{\theta}{2}$，則 $\cos 2\theta$ 以 t 表之為____________ .

解答 $\dfrac{1-6t^2+t^4}{1+2t^2+t^4}$

解析 $\cos 2\theta=2\cos^2\theta-1=2\left(\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)^2-1=2\left(\dfrac{1-t^2}{1+t^2}\right)^2-1=\dfrac{1-6t^2+t^4}{1+2t^2+t^4}$.

22.考慮函數 $f(x)=\cos 2x+4\sin^2 x-\cos x-2$，解方程式 $f(x)=0$，得其解為________________ .

解答 $x=2n\pi\pm\frac{\pi}{3}$ 或 $x=(2n+1)\pi$， n 為整數

解析 $\cos 2x+4\sin^2 x-\cos x-2=0\Rightarrow 2\cos^2 x-1+4\left(1-\cos^2 x\right)-\cos x-2=0$

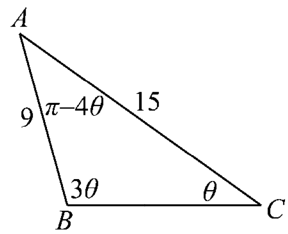
$\therefore 2\cos^2 x+\cos x-1=0 \Rightarrow \therefore (2\cos x-1)(\cos x+1)=0$，$\therefore \cos x=\frac{1}{2}$或$-1$

$\therefore x=2n\pi\pm\frac{\pi}{3}$或$x=(2n+1)\pi$， n為整數．

23.△ABC中，若$\overline{AB}=9$，$\overline{AC}=15$，$\angle ABC=3\angle ACB$，則$\overline{BC}$邊之長為＿＿＿＿＿＿＿．

解答　$4\sqrt{6}$

解析　設$\angle C=\theta$，$\angle B=3\theta$，$\angle A=\pi-4\theta$



由正弦定理知：$\frac{9}{\sin\theta}=\frac{15}{\sin 3\theta}=\frac{15}{3\sin\theta-4\sin^3\theta}$

$\Rightarrow 27\sin\theta-36\sin^3\theta=15\sin\theta \Rightarrow \sin^2\theta=\frac{1}{3}$（$\because \sin\theta\neq 0$）

$\Rightarrow \cos 2\theta=1-2\sin^2\theta=\frac{1}{3}$， $\cos 4\theta=2\cos^2 2\theta-1=-\frac{7}{9}$

$\therefore \overline{BC}^2=9^2+15^2-2\cdot 9\cdot 15\cos(\pi-4\theta)\ =81+225-2\cdot 9\cdot 15\cdot\frac{7}{9}=96$，$\therefore \overline{BC}=\sqrt{96}=4\sqrt{6}$ ．

24.$0\le x<2\pi$， $f(x)=\frac{\sin 2x}{1+\sin x+\cos x}$的(1)最大值=＿＿＿＿＿＿＿，(2)最小值=＿＿＿＿＿＿＿．

解答　(1)$\sqrt{2}-1$;(2)$-\sqrt{2}-1$

解析　令$\sin x+\cos x=t$， $-\sqrt{2}\le t\le\sqrt{2}$

$\therefore t^2=1+2\sin x\cos x=1+\sin 2x \Rightarrow t^2-1=\sin 2x \Rightarrow f(t)=\frac{t^2-1}{1+t}=t-1$

①當$t=\sqrt{2}$時， $f(x)$的最大值為$\sqrt{2}-1$

②當$t=-\sqrt{2}$時， $f(x)$的最小值為$-\sqrt{2}-1$ ．

25.設$\sin\theta$和$\cos\theta$為$4x^2+2x+1=0$之兩根，則$2\sin^2\frac{\theta}{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2=$＿＿＿＿＿＿＿．

解答　$\frac{7}{4}$

解析　由根與係數關係得$\sin\theta+\cos\theta=-\frac{1}{2}$， $\sin\theta\cdot\cos\theta=\frac{1}{4}$

原式$=2\sin^2\frac{\theta}{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2=(1-\cos\theta)(1-\sin\theta)\ =1-(\sin\theta+\cos\theta)+\sin\theta\cdot\cos\theta$

$=1-\left(-\frac{1}{2}\right)+\frac{1}{4}=\frac{7}{4}$ ．